SETS WITH HOLES ON THE REAL LINE: A CONSTRUCTION AND AN APPLICATION

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ABSTRACT

An increasing sequence of closed subsets of the unit interval is constructed, such that no set in the sequence is essentially covered by a finite union of translates of the set preceding it. The result is applied to construct a pathological right-translation-invariant ideal of the Banach lattice L^∞ on R_+.

1. **Introduction.** In [3] a conjecture on the existence of very "thin" measurable sets on the real line was settled in the affirmative by the trick of constructing certain sets with evenly scattered holes. Since both the size and the distribution of the holes could be closely adapted to the purpose in hand, it was tempting to put such sets to further uses. This note illustrates another such use.

The question dealt with in [3] originated in a problem of the theory of function spaces with translations on the positive half-line, as developed in [2] (cf. [1], Chapter 2); so does the question discussed here. In Section 3 we describe this application without technical dependence on those references.

2. **The construction.** R denotes the real line and μ is Lebesgue measure on R. If F, $F' \subset R$ are μ-measurable, we say that F is *essentially contained* in F' if $\mu(F \setminus F') = 0$.

THEOREM 1. *There exists a sequence* (E_m) *of closed subsets of* $[0,1]$ *such that* $\mu(E_1) > 0$, $E_m \subset E_{m+1}$, *and* E_{m+1} *is not essentially contained in any finite union of translates of* E_m, $m = 1,2,\cdots$.

Proof. 1. We define a positive-integral-valued function $p(m;n)$ on the pairs of positive integers, by recursion in the first variable, thus:

$$p(1;n) = n+1, \quad n = 1,2,\cdots; \tag{1}$$

if $p(m;n)$ is known for a given m and all n we define the following auxiliary function by recursion in j:

$$q(m,0;n) = n; \; q(m,j+1;n) = q(m,j;n) + 1 + p(m;q(m,j;n)+1), \quad j = 0,\cdots,n-1, \; n = 1,2,\cdots \tag{2}$$

and finally set

Received September 6, 1966.

(3) $$p(m+1;n) = q(m,n;n) - n + 2.$$

It is immediately verified by induction that $p(m;n)$ is positive-valued and that

(4) $p(m;n)$ is strictly increasing in both variables.

2. For every positive integer m, we define the closed set

$$A_m = R \setminus \bigcup_{n=1}^{\infty} \bigcup_{r=-\infty}^{\infty} (2^{-n}r + (0, 2^{-n-p(m;n)})).$$

The open intervals $2^{-n}r + (0, 2^{-n-p(m;n)})$, $r = 0, \pm 1, \pm 2, \cdots$, are the *holes of order n* of A_m; the images of these holes under the translation $t \to t' + t$ are the *holes of order n* of $t' + A_m$. On account of (4), $A_m \subset A_{m+1}$ for all m.

We set $E_m = [0,1] \cap A_m$, so that (E_m) is a sequence of closed subsets of $[0,1]$ with $E_m \subset E_{m+1}$. Using (1), we find

$$\mu(E_1) \geqq 1 - \sum_{n=1}^{\infty} \sum_{r=0}^{2^n-1} 2^{-n-p(1;n)} = 1 - \sum_{n=1}^{\infty} 2^{-n-1} = \tfrac{1}{2} > 0.$$

To conclude the proof, we show that, for any given m, E_{m+1} is not essentially contained in any finite union of translates of A_m, let alone of E_m.

3. More specifically, we claim that for any given positive integers m, k and any $t_1, \cdots, t_k \in R$ we have

(5) $$\mu([1-2^{-k},1] \cap E_{m+1}) > \mu([1-2^{-k},1] \cap \bigcup_{i=1}^{k} (t_i + A_m)),$$

which implies the desired conclusion.

Indeed, since $p(m+1;n) \geqq p(1;1) = 2$ for all n, $[1-2^{-k},1]$ does not meet any holes of A_{m+1} of any order strictly smaller than k; and it contains, for each $n \geqq k$, 2^{n-k} holes of order n and total measure $2^{n-k}2^{-n-p(m+1;n)} = 2^{-k-p(m+1;n)}$ $\leqq 2^{-n-p(m+1;k)}$ (on account of (4)). Therefore

(6) $$\mu([1-2^{-k},1] \times E_{m+1}) = \mu([1-2^{-k},1] \cap A_{m+1})$$
$$\geqq 2^{-k} - \sum_{n=k}^{\infty} 2^{-n-p(m+1;k)} = 2^{-k}(1 - 2^{1-p(m+1;k)}).$$

On the other hand, $[1-2^{-k},1]$, being of length 2^{-k}, contains at least one interval of the form $(t_1 + 2^{-k-1}r, t_1 + 2^{-k-1}(r+1))$ (r an integer), and therefore contains one complete hole of $t_1 + A_m$ of order $k+1$ and length $2^{-(k+1)-p(m;k+1)}$ $= 2^{-q(m,1;k)}$; this hole again contains a complete hole of $t_2 + A_m$ of order $q(m,1;k)+1$ and length $2^{-q(m,2;k)}$ (we here use (2)); continuing in this fashion, we find that $[1-2^{-k},1]$ contains an open interval of length $2^{-q(m,k;k)}$ that is a hole of $t_k + A_m$ contained in a hole of $t_{k-1} + A_m \cdots$ contained in a hole of $t_1 + A_m$, and thus disjoint from $\bigcup_{i=1}^{k}(t_i + A_m)$. Therefore, using (3),

$$\mu([1-2^{-k}] \cap \bigcup_{i=1}^{k} (t_i + A_m)) \leqq 2^{-k} - 2^{-q(m,k;k)} = 2^{-k}(1 - 2^{2-p(m+1;k)}).$$

This, together with (6), implies (5), thus establishing our claim.

3. **The application.** R_+ denotes the positive real half-line. *Measure*, *measurable*, *null*, all refer to Lebesgue measure. L^∞ is the usual Banach lattice of (equivalence classes modulo null sets of) essentially bounded measurable real valued functions on R_+, with the essential-supremum norm $\|\cdot\|$. A constant in L^∞ is denoted by its constant value, the characteristic function of the measurable set $E \subset R_+$ by χ_E.

For $f \in L^\infty$, $s \in R$, we define the translate $T_s f \in L^\infty$ by

$$T_s f(t) = \begin{cases} f(t-s) & t \geqq \max\{0, s\} \\ 0 & 0 \leqq t < s \end{cases}$$

(with the usual measure-theoretical grain of salt). $F \subset L^\infty$ is an *ideal* in L^∞ if F is a linear manifold satisfying

$$\text{(F)}: f \in F,\ g \in L^\infty,\ |g| \leqq |f| \text{ implies } g \in F.$$

An ideal $F \subset L^\infty$ is *right-translation-invariant* [*translation-invariant*] if it satisfies

$$\text{(RT) [(T)]}: f \in F,\ s \in R_+[s \in R] \text{ implies } T_s f \in F.$$

It is an easy exercise to show that the L^∞-closure of a right-translation-invariant ideal is another (the same is true, for that matter, for translation-invariant ideals). If F is a right-translation-invariant ideal, the smallest translation-invariant ideal containing F is, clearly, $T^- F = \{T_{-s} f : f \in F, s \in R_+\}$.

THEOREM 2. *There exists a closed right-translation-invariant ideal F in L^∞ such that $T^- F$ is not closed in L^∞.*

Proof. Let (E_m) be as in Theorem 1, and set $B = \bigcup_{m=1}^{\infty}(m + E_m)$, a closed set in R_+. Define G as the set of those $f \in L^\infty$ that vanish a.e. outside some finite union of right-translates of B; thus $f \in L^\infty$ with $\|f\| \leqq 1$ is in G if and only if $|f| \leqq \chi_C$, where $C = \bigcup_{i=1}^{k}(s_i + B)$ for some positive integer k and numbers $s_i \in R_+$, $i = 1, \cdots, k$; in particular, if $E \subset R_+$ is measurable, $\chi_E \in G$ if and only if E is essentially contained in such a set C.

We claim that this is specifically not the case for $E = p + E_{p+1}$ for any positive integer p; otherwise, indeed, $p + E_{p+1}$ would be essentially contained in $C = \bigcup_{m=1}^{\infty} \bigcup_{i=1}^{k}(m + s_i + E_m)$; since $(p + E_{p+1}) \cap (m + s_i + E_m)$ is null for

$m \geqq p+1$, and $E_m \subset E_p$ for $m \leqq p$, E_{p+1} would—in R of course—be essentially contained in $\bigcup_{m=1}^{p} \bigcup_{i=1}^{k}(-p+m+s_i+E_p)$, contradicting the conclusion of Theorem 1.

G is obviously a right-translation-invariant ideal, so that its L^∞-closure, which we choose as our F, is another.

Now $\chi_{m+E_m} \in G$, so that $\chi_{E_m} = T_{-m}\chi_{m+E_m} \in T^-F$, $m = 1,2,\cdots$, with $\| \chi_{E_m} \| = 1$. Therefore $(\sum_{m=1}^{n} 2^{-m}\chi_{E_m})$ is an L^∞-Cauchy sequence in T^-F, and its L^∞-limit is $f_0 = \sum_{m=1}^{\infty} 2^{-m}\chi_{E_m} \leqq \chi_{[0,1]}$. We claim that f_0 is not in T^-F, thus proving that T^-F is not L^∞-closed.

For assume that $f_0 \in T^-F$, i.e., $f_0 = T_{-s}f$ for some $f \in F$, $s \in R_+$; choose a positive integer $p \geqq s$, and set $E = p + E_{p+1}$; then $0 \leqq T_s f_0 = T_s T_{-s} f \leqq |f|$, and therefore

$$0 \leqq 2^{-p-1}\chi_E = T_p(2^{-p-1}\chi_{E_{p+1}}) \leqq T_p f_0 = T_{p-s}T_s f_0 \leqq T_{p-s}|f| \in F,$$

whence $\chi_E \in F$. There exists, therefore, $g \in G$ such that $|\chi_E - g| \leqq \|\chi_E - g\| \leqq \frac{1}{2}$, whence $\chi_E - \chi_E g \leqq \chi_E |\chi_E - g| \leqq \frac{1}{2}\chi_E$, which implies $|g| \geqq \chi_E g \geqq \frac{1}{2}\chi_E \geqq 0$. Thus $\chi_E \in G$; and this was shown above to be impossible.

Remark. Theorem 2 provides a negative answer to the question in [2], p. 237 (following Theorem 4.6) and justifies the statement in [1], p. 61 (following 23.H); it is sufficient to provide F and T^-F with the norm of L^∞.

Our proof of Theorem 2 used the "local structure" of R_+ very strongly, via Theorem 1. This prompted the author to raise the question concerning the analogous problem for the sequence space l^∞ (on the set Z_+ of non-negative integers). The referee has indicated the proof of the following analogue of Theorem 2; it may be adapted in an obvious way to yield an alternative proof of Theorem 2 itself. The meaning of the terms will be clear without fresh definitions.

Theorem. 3. *There exists a closed right-translation-invariant ideal F in l^∞ such that $T^\infty F^-$ is not closed in l^∞.*

Proof. Let a_n be the product of the first n primes and set $E_n = \{a_n^h : h = 1,2,\cdots\} \subset Z_+$, $n = 1,2,\cdots$. Let G be the set of those $f \in l^\infty$ that vanish outside some finite union of right-translates of $1+E_1$, $2+E_2,\cdots$, and let F be the l^∞-closure of G. F if a closed right-translation-invariant ideal.

As in the proof of Theorem 2, $(\sum_{m=1}^{n} 2^{-m}\chi_{E_m})$ is an l^∞-Cauchy sequence in T^-F; if its l^∞-limit $f_0 = \sum_{m=1}^{\infty} 2^{-m}\chi_{E_m}$ were in T^-F, it would follow as in that proof that $\chi_{p+E_{p+1}} \in G$ for some $p \in Z_+$; but this means that $p + E_{p+1} \subset \bigcup_{i=1}^{k}(r_i + i + E_i)$ for some positive integer k and $r_i \in Z_+$, $i = 1,\cdots,k$. This, however, is impossible, for $(p+E_{p+1}) \cap (r_i + i + E_i)$ is a finite set for each i: indeed, if $p + a_{p+1}^h = r_i + i + a_i^{h'}$, the integer $|r_i + i - p|$ is divisible by a_q^j, where $q = \min\{i, p+1\}$, $j = \min\{h, h'\}$; thus either h or h' is bounded, and the intersection is consequently finite, unless $p = r_i + i$; but in this last case, $(p + E_{p+1}) \cap (r_i + i + E_i) = p + (E_{p+1} \cap E_i)$ is empty, since $i \leqq r_i + i = p < p + 1$.

The author wishes to thank the referee for suggesting the inclusion of the preceding result. Part of the research in this paper was carried out while the author was visiting professor at Carnegie Institute of Technology, Pittsburgh, U.S.A.

References

1. J. L. Massera and J. J. Schäffer, *Linear differential equations and function spaces*, Academic Press, New York, 1966.

2. J. J. Schäffer, *Function spaces with translations*, Math. Ann. **137** (1959), 209–262.

3. ———, *Proof of a conjecture about measurable sets on the real line*, Proc. Amer. Math. Soc. **13** (1962), 134–135.

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